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BAYESIAN AND E –BAYESIAN ESTIMATION OF THE UNKNOWN SHAPE PARAMETER OF EXPONENTIATED INVERTED WEIBULL DISTRIBUTION USING DIFFERENT LOSS FUNCTIONS

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ABSTRACT

The present paper is concerned with using Bayesian and E-Bayesian method of estimation to find estimates for the shape parameter of Exponentiated Inverted Weibull distribution. These estimators are derived by using different loss functions. In this paper, Bayesian estimates are derived by using informative prior.

KEY WORDS: Exponentiated Inverted Weibull distribution, Bayes estimates, E-Bayes estimates, Degroot loss function, Al-Bayyati loss function.

1. Introduction

The Inveted Weibull distribution is one of the most widely used distribution for analyzing life time data. It is found to be useful in diverse fields ranging from Engineering to Medical Science. The two parameter Exponentiated Inverted Weibull distribution (EIWD) has been proposed by Flaih et al (2012) and is a generalization to the Inverted Weibull distribution through adding a new shape parameter $\theta \in R^{+}$ by exponentiation to distribution function F. The cumulative distribution function (c. d. f) of EIWD is given by

$$F_{\theta}(x) = \left(e^{-x^{-\beta}}\right)^{\theta} \quad ; \quad x,\ \theta,\ \beta > 0$$

which is simply the θth power of the distribution function of the Standard Inverted Weibull distribution. Here θ and β are the shape parameters of EIWD.

The probability density function (p.d.f) of EIWD with two shape parameters θ and β is given by

$$f\,((x,\theta,\beta) = \theta\,\beta\; x^{-(\beta+1)}\, e^{-\theta\, x^{-\beta}} \;;\; x > 0,\ \theta > 0, \beta > 0 \qquad (5.1.1)$$

The Exponentiated Inverted Weibull distribution also has a physical interpretation. If there are m-components in a parallel system and the life times of the components are independent and identically distributed (i.i.d) as exponentiated inverted weibull distribution, then the system lifetime variable also has exponentiated inverted weibull distribution. Many characteristics of Exponentiated Weibull Distribution has been found in the literature. Soland (1968) discusses the Bayesian analysis of the Weibull Process. Mudholka et al (1995) introduced the exponentiated weibull distribution as a generalization of the standard weibull distribution and as a suitable model for bus motor failure time data. Mudholkar and Huston (1996) applied the exponentiated weibull distribution to the flood data with some properties. Ahmad et al (2015) discusses the Bayesian estimator of shape parameter of exponentiated inverted weibull distribution under different loss function namely square error loss function, entropy loss function and precautionary loss function. El-Din et al (2014) worked on statistical inference and prediction for the inverse weibull distribution based on record data. Aljouharah (2013) estimates the parameters of EIWD under type-II censoring.

Bayesian estimation approach has received great attention by the researchers and they had proved that Bayes estimate perform better than classical estimators. A new approach of Bayesian estimation called Expected Bayesian or E-Bayesian method has been introduced by Han (2007). He obtained the E- Bayes estimate of failure probability by considering quadratic loss function and discussed the properties of E-Bayes estimate and showed that E-Bayes estimate is efficient and easy to operate. [Okasha (2012)] constructed Bayesian and E- Bayesian method for estimating the scale parameter, reliability and hazard function of the Weibull Distribution under type-II censored samples by considering the squared error loss function. [Jaheen and Okasha (2011)] compared the Bayesian and E- Bayesian estimator for the parameters and reliability function of the Burr – XII model under type-II censoring by considering the squared error and linex loss function. They have shown that the performance of E-Bayes estimate is better. [Azimi *et al*, (2013)] studied E-Bayesian estimation of generalized half logistic progressive type-ii censored data. [Reyad *et al*, (2015)] develop E-Bayesian estimators of parameter of the Gumbel type - II distribution.

The main objective of this paper is to make comparison between the Bayesian and E-Bayesian method for estimating unknown shape parameter of Exponentiated Inverted Weibull Distribution. The Bayes and E-Bayes estimators are obtained by using three different loss functions under conjugate prior distribution. Properties of E-Bayes estimates are also studied.

In this chapter, Bayes and E-Bayes approaches have been used for obtaining the estimates of the unknown parameter of EIWD. In section 2, Bayes estimator are derived by using informative prior under Degroot, Al-Bayyati and Minimum Expected loss function. In Section 3, Expected – Bayes estimators are derived based on conjugate prior for the parameter of interest. Properties of E-Bayesian estimates are discussed in Section 4.

2. Bayesian Estimation

In this section, Bayesian estimators of the unknown shape parameter of EIWD are obtained by using gamma informative prior under three different loss functions. Let a random sample of size n be drawn from EIWD with pdf given by eq. (1.1), then the likelihood function is given by:-

$$L(\underline{x}, \theta, \beta) = \prod_{i=1}^{n} f(x_i, \theta, \beta) ;$$

where $\underline{x} = (x_1, x_2, x_3, x_4 \ldots\ldots\ldots x_n)$ and θ is unknown and β is known parameter.

$$= (\theta\beta)^n \prod_{i=1}^{n} x_i^{-(\beta+1)} e^{-\theta \sum_{i=1}^{n} x_i^{-\beta}} \tag{2.1}$$

Here, we are using the gamma distribution as an conjugate prior distribution of θ .The gamma prior distribution with shape parameter and scale parameter c and r respectively, has the following pdf

$$g(\theta/c,r) = \frac{r^c}{\Gamma c} \theta^{c-1} e^{-\theta r} ; \theta > 0, c, r > 0 \tag{2.2}$$

This prior distribution was first used by Papadopoulos (1978).

On combining (2.1) and (2.2), and using Bayes theorem, the posterior density of θ given $\underline{x}$ is given by

$$P\left(\theta/\underline{x}\right) \ \alpha \ (\theta\beta)^n \prod_{i=1}^{n} x_i^{-(\beta+1)} e^{-\theta \sum_{i=1}^{n} x_i^{-\beta}} * \frac{r^c}{\Gamma c} \theta^{c-1} e^{-\theta r}$$

$$= k \ \theta^{n+c-1} e^{-\theta \left[\sum_{i=1}^{n} x_i^{-\beta} + r\right]}$$

where k is independent of θ.

Thus, Posterior density is given by

$$P\left(\theta/x\right) = \frac{\left(\sum_{i=1}^{n} x_i^{-\beta} + r\right)^{n+c}}{\Gamma\, n+c} \theta^{n+c-1} e^{-\theta \left[\sum_{i=1}^{n} x_i^{-\beta} + r\right]} ; \quad \theta > 0, c, r > 0 \tag{2.3}$$

Bayesian estimate of θ under De-groot loss function using gamma prior

If $\hat{\theta}$ is an estimator of θ then the De-groot loss function is given by

$$L\left(\theta, \hat{\theta}\right) = \frac{\theta - \hat{\theta}}{\hat{\theta}} .$$

For more details about De-groot loss function, one may refer Degroot (1970), optimal statistical decision.

The Bayesian estimate using De-groot loss function is given by

$$\hat{\theta}_{DG} = \frac{E\left(\theta^2/\underline{x}\right)}{E\left(\theta/\underline{x}\right)}$$

Now, $E\left(\theta/\underline{x}\right) = \frac{n+c}{\left(\sum_{i=1}^{n} x_i^{-\beta} + r\right)}$.

$$E\left(\theta^2/\underline{x}\right) = \frac{(n+c+1)\ (n+c)}{\left(\sum_{i=1}^{n} x_i^{-\beta} + r\right)^2} .$$

Bayesian estimator under De-groot loss function is given by

$$\hat{\theta}_{DG} = \frac{E\left(\theta^2/\underline{x}\right)}{E\left(\theta/\underline{x}\right)} =$$

$$= \frac{(n+c+1)}{\left(\sum_{i=1}^{n} x_i^{-\beta} + r\right)}. \tag{2.4}$$

Bayesian estimate of θ under Al-Bayyati new loss function using gamma prior :-

Al-Bayyati (2002) introduced a new loss function given by

$$L\left(\theta,\hat{\theta}\right) = \theta^{c_1}\ \left(\hat{\theta}-\theta\right)^2 \ ;\ c_1\ \varepsilon\ \mathrm{R}\ .$$

The Al-Bayyati loss function introduces the additional parameter c_1 , which assists in determining a flatter loss function and it specifically generalizes the Square Error loss function. c_1 can also be considered the order of weighting of the quadratic component.

$$E\left(L\left(\theta,\hat{\theta}\right)\right) = \int_0^{\infty} \theta^{c_1}\ \left(\hat{\theta}-\theta\right)^2 * P\left(\theta/x\right) d\theta$$

$$= \hat{\theta}^2 * \frac{\Gamma\,(n+c+c_1)}{\Gamma\,(n+c)\ \left(\sum_{i=1}^{n} x_i^{-\beta}+r\right)^{c_1}} + \frac{\Gamma\ (n+c+c_1+2)}{\Gamma\,(n+c)\left(\sum_{i=1}^{n} x_i^{-\beta}+r\right)^{c_1+2}} - 2\hat{\theta}\ \frac{\Gamma\ (n+c+c_1+1)}{\Gamma\,(n+c)\left(\sum_{i=1}^{n} x_i^{-\beta}+r\right)^{c_1+1}}$$

Bayes estimator under Al-Bayyati loss function is

$$\hat{\theta}_{AB} = \frac{(n+c+c_1)}{\left(\sum_{i=1}^{n} x_i^{-\beta} + r\right)} \tag{2.5}$$

Bayesian estimate of θ under Minimum Expected loss function using gamma prior :-

Tummala and Sathe (1978) proposed minimum expected loss function (MELF) which is given as

$L\left(\theta,\hat{\theta}\right) = \frac{\left(\hat{\theta}-\theta\right)^2}{\theta^2}$; where $\hat{\theta}$ is an estimator of θ.

$$E\left(L\left(\theta,\hat{\theta}\right)\right) = \int_0^{\infty} \frac{\left(\hat{\theta}-\theta\right)^2}{\theta^2} * P\left(\theta/x\right) d\theta$$

$$= \frac{\left(\sum_{i=1}^{n} x_i^{-\beta} + r\right)^2\ \hat{\theta}^2}{(n+c-1)\ (n+c-2)} + 1 - \frac{2\hat{\theta}\ \left(\sum_{i=1}^{n} x_i^{-\beta} + r\right)}{(n+c-1)}$$

Bayes estimator under Minimum expected loss function is given by

$$\hat{\theta}_{ME} = \frac{(n+c-2)}{\left(\sum_{i=1}^{n} x_i^{-\beta} + r\right)} \tag{2.6}$$

3. E-Bayesian Estimates

According to Han (1997), the prior parameters c and r should be selected to guarantee that the prior g(θ /c, r) in (2.2) be a decreasing function of θ. The derivative g(θ /c, r) with respect to θ is

$$\frac{d}{d\theta}\ g(\theta/c,r) = \frac{r^c}{\Gamma c}\ \theta^{c-2}\ e^{-\theta r}\ [\,(c-1) - r\theta\,]$$

Thus, for $0 < c < 1$ and $r > 0$, the prior distribution $g(\theta / c, r)$ is a decreasing function of θ.

The E – Bayesian estimate of θ i.e. expectation of the Bayesian estimate of θ is given by

$$\hat{\theta}_{EB} = E\left(\theta / \underline{x}\right)$$

$$= \iint_Q \hat{\theta}_{BE}\, \pi(\theta, c, r) * dr\, dc\ ;$$

where Q is the domain of c and r for which the prior density is decreasing in θ. $\hat{\theta}_{BE}$ is the Bayesian estimate of θ under the different loss function. For more details, one can refer Han (2009).

In order to obtain E-Bayesian estimates of θ, we have to choose prior distribution of hyper parameter c and r. These distributions are used to study the impact of different prior distributions on E-Bayesian estimation of θ. The following distributions of c and r are given by

$$\pi_1(\theta, c, r) = \frac{2\,(s-r)}{s^2}\ ;\ 0 < c < 1,\ 0 < r < s \tag{3.1}$$

$$\pi_2(\theta, c, r) = \frac{1}{s}\ ;\ 0 < c < 1,\ 0 < r < s \tag{3.2}$$

$$\pi_3(\theta, c, r) = \frac{2\,r}{s^2}\ ;\ 0 < c < 1,\ 0 < r < s \tag{3.3}$$

3(a) E-Bayesian Estimation of θ under De-groot loss function

E-Bayesian estimates of θ relative to De-groot loss function based on $\pi_1(\theta, c, r)$, is denoted by $\hat{\theta}_{EBDG_1}$ and is obtained by using (2.4) and (3.1).

$$\hat{\theta}_{EBDG_1} = \int_0^1 \int_0^s \frac{(n + c + 1)}{(\sum_{i=1}^n x_i^{-\beta} + r)} * \frac{2\,(s-r)}{s^2}\, dr\, dc$$

$$= \frac{2}{s^2}\left[\int_0^1 (n + c + 1)\left(\int_0^s \frac{(s-r)}{(\sum_{i=1}^n x_i^{-\beta} + r)}\, dr\right) dc\ \right] \tag{3.4}$$

$$\int_0^s \frac{(s-r)}{(\sum_{i=1}^n x_i^{-\beta} + r)}\, dr = \left[log\left(\frac{(s + \sum_{i=1}^n x_i^{-\beta})}{\sum_{i=1}^n x_i^{-\beta}}\right)\right]\left(s + \sum_{i=1}^n x_i^{-\beta}\right) - s \tag{3.5}$$

On using (3.5) in (3.4), we get

$$\hat{\theta}_{EBDG_1} = \frac{2n+3}{s^2} * \left\{\left[log\left(\frac{(s + \sum_{i=1}^n x_i^{-\beta})}{\sum_{i=1}^n x_i^{-\beta}}\right)\right]\left(s + \sum_{i=1}^n x_i^{-\beta}\right) - s\ \right\} \tag{3.6}$$

E-Bayesian estimates of θ relative to De-groot loss function based on $\pi_2(\theta, c, r)$, is denoted by $\hat{\theta}_{EBDG_2}$ and is obtained by using (2.4) and (3.2).

$$\hat{\theta}_{EBDG_2} = \int_0^1 \int_0^s \frac{(n + c + 1)}{(\sum_{i=1}^n x_i^{-\beta} + r)} * \frac{1}{s}\, dr\, dc$$

$$= \frac{1}{s} * \left[log\left(\frac{(s + \sum_{i=1}^n x_i^{-\beta})}{\sum_{i=1}^n x_i^{-\beta}}\right)\right] * \left(\frac{2n+3}{2}\right) \tag{3.7}$$

E-Bayesian estimates of θ relative to De-groot loss function based on $\pi_3(\theta, c, r)$, is denoted by $\hat{\theta}_{EBDG_3}$ and is obtained by using (2.4) and (3.3).

$$\hat{\theta}_{EBDG_3} = \int_0^1 \int_0^s \frac{(n + c + 1)}{(\sum_{i=1}^n x_i^{-\beta} + r)} * \frac{2r}{s^2} \, dr \, dc$$

$$= \int_0^1 \frac{2(n+c+1)}{s^2} \left(\int_0^s \frac{r}{(\sum_{i=1}^n x_i^{-\beta} + r)} \, dr \right) dc \quad (3.8)$$

$$\int_0^s \frac{r}{(\sum_{i=1}^n x_i^{-\beta} + r)} dr = s - \sum_{i=1}^n x_i^{-\beta} \left[log \left(\frac{(s + \sum_{i=1}^n x_i^{-\beta})}{\sum_{i=1}^n x_i^{-\beta}} \right) \right] \quad (3.9)$$

On using (3.9) in (3.8), we get

$$\hat{\theta}_{EBDG_3} = \int_0^1 \frac{2(n+c+1)}{s^2} \left\{ s - \sum_{i=1}^n x_i^{-\beta} \left[log \left(\frac{(s + \sum_{i=1}^n x_i^{-\beta})}{\sum_{i=1}^n x_i^{-\beta}} \right) \right] \right\} dc$$

$$= \frac{(2n+3)}{s^2} \left\{ s - \sum_{i=1}^n x_i^{-\beta} \left[log \left(\frac{(s + \sum_{i=1}^n x_i^{-\beta})}{\sum_{i=1}^n x_i^{-\beta}} \right) \right] \right\} \quad (3.10)$$

3(b) E-Bayesian estimation under Al-Bayyati loss function.

E-Bayesian estimates of θ relative to Al-Bayyati loss function based on $\pi_1(\theta, c, r)$, is denoted by $\hat{\theta}_{EBAB_1}$ and is obtained by using (2.5) and (3.1).

$$\hat{\theta}_{EBAB_1} = \int_0^1 \int_0^s \frac{(n + c + c_1)}{(\sum_{i=1}^n x_i^{-\beta} + r)} * \frac{2(s-r)}{s^2} \, dr \, dc$$

$$= \left\{ \left[log \left(\frac{(s + \sum_{i=1}^n x_i^{-\beta})}{\sum_{i=1}^n x_i^{-\beta}} \right) \right] \left(s + \sum_{i=1}^n x_i^{-\beta} \right) - s \right\} * \frac{1}{s^2} (2n + 1 + 2c_1) \quad (3.11)$$

E-Bayesian estimates of θ relative to Al-Bayyati loss function based on $\pi_2(\theta, c, r)$, is denoted by $\hat{\theta}_{EBAB_2}$ and is obtained by using (2.5) and (3.2).

$$\hat{\theta}_{EBAB_2} = \int_0^1 \int_o^s \frac{(n + c + c_1)}{(\sum_{i=1}^n x_i^{-\beta} + r)} * \frac{1}{s} \, \mathrm{dr} \, \mathrm{dc}$$

$$= \left[\log \left(\frac{s + \sum_{i=1}^n x_i^{-\beta}}{\sum_{i=1}^n x_i^{-\beta}} \right) \right] * \frac{(2n+1+2c_1)}{2s} \quad (3.12)$$

E-Bayesian estimates of θ relative to Al-Bayyati loss function based on $\pi_3(\theta, c, r)$, is denoted by $\hat{\theta}_{EBAB_3}$ and is obtained by using (2.5) and (3.3).

$$\hat{\theta}_{EBAB_3} = \int_0^1 \int_o^s \frac{(n + c + c_1)}{(\sum_{i=1}^n x_i^{-\beta} + r)} * \frac{2r}{s^2} \, \mathrm{dr} \, \mathrm{dc}$$

$$= \frac{2}{s^2} * \int_0^1 (n + c + c_1) \left(\int_o^s \frac{r + \sum_{i=1}^n x_i^{-\beta} - \sum_{i=1}^n x_i^{-\beta}}{(\sum_{i=1}^n x_i^{-\beta} + r)} \, dr \right) \mathrm{dc} \quad (3.13)$$

$$\int_o^s \frac{r + \sum_{i=1}^n x_i^{-\beta} - \sum_{i=1}^n x_i^{-\beta}}{(\sum_{i=1}^n x_i^{-\beta} + r)} dr = s - \sum_{i=1}^n x_i^{-\beta} \left[\log \left(\frac{s + \sum_{i=1}^n x_i^{-\beta}}{\sum_{i=1}^n x_i^{-\beta}} \right) \right] \quad (3.14)$$

On using (3.14) in (3.13), we have

$$\hat{\theta}_{EBAB_3} = \frac{(2n+1+2c_1)}{s^2} \left\{ s - \sum_{i=1}^n x_i^{-\beta} \left[\log \left(\frac{s + \sum_{i=1}^n x_i^{-\beta}}{\sum_{i=1}^n x_i^{-\beta}} \right) \right] \right\} \quad (3.15)$$

3(c) E-Bayesian estimation under Minimum Expected loss function.

E - Bayesian estimates of θ relative to Minimum expected loss function based on $\pi_1(\theta,c,r)$, is denoted by $\hat{\theta}_{EBME_1}$ and is obtained by using (2.6) and (3.1).

$$\hat{\theta}_{EBME_1} = \int_0^1\int_0^s \frac{(n+c-2)}{(\sum_{i=1}^{n} x_i^{-\beta} + r)} * \frac{2\,(s-r)}{s^2}\, dr\, dc$$

$$= \int_0^1 \frac{2\,(n+c-2)}{s^2}\left(\int_0^s \frac{(s-r)}{(\sum_{i=1}^{n} x_i^{-\beta} + r)}\, dr\right) dc$$

$$= \left\{\left[\log\left(\frac{(s+\sum_{i=1}^{n} x_i^{-\beta}}{\sum_{i=1}^{n} x_i^{-\beta}}\right)\right]\left(s+\sum_{i=1}^{n} x_i^{-\beta}\right) - s\right\} * \frac{2}{s^2} * \frac{2n-3}{2} \tag{3.16}$$

E-Bayesian estimates of θ relative to Minimum expected loss function based on $\pi_2(\theta,c,r)$, is denoted by $\hat{\theta}_{EBME_2}$ and is obtained by using (2.6) and (3.2).

$$\hat{\theta}_{EBME_2} = \int_0^1\int_o^s \frac{(n+c-2)}{(\sum_{i=1}^{n} x_i^{-\beta} + r)} * \frac{1}{s}\, \mathrm{dr\ dc}$$

$$= \left[\log\left(\frac{s+\sum_{i=1}^{n} x_i^{-\beta}}{\sum_{i=1}^{n} x_i^{-\beta}}\right)\right] * \frac{(2n-3)}{2s} \tag{3.17}$$

E-Bayesian estimates of θ relative to Minimum expected loss function based on $\pi_3(\theta,c,r)$, is denoted by $\hat{\theta}_{EBME_3}$ and is obtained by using (2.6) and (3.3).

$$\hat{\theta}_{EBME_3} = \int_0^1\int_o^s \frac{(n+c-2)}{(\sum_{i=1}^{n} x_i^{-\beta} + r)} * \frac{2r}{s^2}\, \mathrm{dr\ \ dc}$$

$$= \frac{2}{s^2} * \int_0^1 (n+c-2)\left(\int_o^s \frac{r+\sum_{i=1}^{n} x_i^{-\beta} - \sum_{i=1}^{n} x_i^{-\beta}}{(\sum_{i=1}^{n} x_i^{-\beta} + r)}\, dr\right) \mathrm{dc}$$

From equation (3.14), we have

$$\int_o^s \frac{r+\sum_{i=1}^{n} x_i^{-\beta} - \sum_{i=1}^{n} x_i^{-\beta}}{(\sum_{i=1}^{n} x_i^{-\beta} + r)}\, dr = s - \sum_{i=1}^{n} x_i^{-\beta}\left[\log\left(\frac{s+\sum_{i=1}^{n} x_i^{-\beta}}{\sum_{i=1}^{n} x_i^{-\beta}}\right)\right]$$

Therefore, we have

$$\hat{\theta}_{EBME_3} = \frac{(2n-3)}{s^2}\left\{s - \sum_{i=1}^{n} x_i^{-\beta}\left[\log\left(\frac{s+\sum_{i=1}^{n} x_i^{-\beta}}{\sum_{i=1}^{n} x_i^{-\beta}}\right)\right]\right\} \tag{3.18}$$

4. Properties of E-Bayesian estimators .

In this section, we discuss the relations among different E-Bayesian estimators obtained

under De-groot loss function , Al-Bayyati loss function and Minimum expected loss function i.e. $\hat{\theta}_{EBDG_i}$, $\hat{\theta}_{EBAB_i}$ and $\hat{\theta}_{EBME_3}$ (i = 1, 2, 3)

Theorem 4.1:- E-Bayes estimator obtained under De-groot loss function follow the following results:-

i. $\hat{\theta}_{EBDG_3} < \hat{\theta}_{EBDG_2} < \hat{\theta}_{EBDG_1}$.

ii. $\lim_{\sum_{i=1}^{n} x_i^{-\beta}\to\infty} \hat{\theta}_{EBDG_1} = \lim_{\sum_{i=1}^{n} x_i^{-\beta}\to\infty} \hat{\theta}_{EBDG_2} = \lim_{\sum_{i=1}^{n} x_i^{-\beta}\to\infty} \hat{\theta}_{EBDG_3}$.

Proof:- From eq.(3.6) and (3.7) we get

$$\hat{\theta}_{EBDG_1} - \hat{\theta}_{EBDG_2} = \frac{2n+3}{s}\left\{log\left(\frac{(s+\sum_{i=1}^{n} x_i^{-\beta}}{\sum_{i=1}^{n} x_i^{-\beta}}\right)*\left(\frac{\sum_{i=1}^{n} x_i^{-\beta}}{s}+\frac{1}{2}\right)-1\right\} \quad (4.1)$$

From equation (3.7) and (3.10), we get

$$\hat{\theta}_{EBDG_2} - \hat{\theta}_{EBDG_3} = \frac{2n+3}{s}\left\{log\left(\frac{(s+\sum_{i=1}^{n} x_i^{-\beta}}{\sum_{i=1}^{n} x_i^{-\beta}}\right)*\left(\frac{\sum_{i=1}^{n} x_i^{-\beta}}{s}+\frac{1}{2}\right)-1\right\} \quad (4.2)$$

From equation (4.1) and (4.2), we get

$$\hat{\theta}_{EBDG_1} - \hat{\theta}_{EBDG_2} = \hat{\theta}_{EBDG_2} - \hat{\theta}_{EBDG_3}$$

$$= \frac{2n+3}{s}\left\{log\left(\frac{(s+\sum_{i=1}^{n} x_i^{-\beta}}{\sum_{i=1}^{n} x_i^{-\beta}}\right)*\left(\frac{\sum_{i=1}^{n} x_i^{-\beta}}{s}+\frac{1}{2}\right)-1\right\} \quad (4.3)$$

For -1 < y < 1, we have

$$\ln(1+y) = y - \frac{y^2}{2} + \frac{y^3}{3} - \cdots$$

Assuming $y = \frac{s}{\sum_{i=1}^{n} x_i^{-\beta}}$; when $0 < \frac{s}{\sum_{i=1}^{n} x_i^{-\beta}} < 1$, we get

$$\frac{2n+3}{s}\left\{log\left(\frac{(s+\sum_{i=1}^{n} x_i^{-\beta}}{\sum_{i=1}^{n} x_i^{-\beta}}\right)*\left(\frac{\sum_{i=1}^{n} x_i^{-\beta}}{s}+\frac{1}{2}\right)-1\right\}$$
$$= \frac{2n+3}{s} * \left\{\left(\frac{s}{\sum_{i=1}^{n} x_i^{-\beta}} - \frac{s^2}{2\left(\sum_{i=1}^{n} x_i^{-\beta}\right)^2} + \frac{s^3}{3\left(\sum_{i=1}^{n} x_i^{-\beta}\right)^3} - \cdots\right) * \left(\frac{\sum_{i=1}^{n} x_i^{-\beta}}{s} + \frac{1}{2}\right) - 1\right\}$$

$$= \frac{2n+3}{s} * \left\{\frac{s^2}{12\left(\sum_{i=1}^{n} x_i^{-\beta}\right)^2} + \frac{s^3}{6\left(\sum_{i=1}^{n} x_i^{-\beta}\right)^3} - \ldots\ldots\ldots\right\} > 0 \quad (4.4)$$

$$\Rightarrow \hat{\theta}_{EBDG_3} < \hat{\theta}_{EBDG_2} < \hat{\theta}_{EBDG_1}.$$

Proof (ii) :- From equation (4.3), we get

$$\hat{\theta}_{EBDG_1} - \hat{\theta}_{EBDG_2} = \hat{\theta}_{EBDG_2} - \hat{\theta}_{EBDG_3} = \frac{2n+3}{s}\left\{log\left(\frac{(s + \sum_{i=1}^{n} x_i^{-\beta}}{\sum_{i=1}^{n} x_i^{-\beta}}\right)*\left(\frac{\sum_{i=1}^{n} x_i^{-\beta}}{s}+\frac{1}{2}\right)-1\right\}$$

$$\lim_{\sum_{i=1}^{n} x_i^{-\beta} \to \infty} \hat{\theta}_{EBDG_1} - \hat{\theta}_{EBDG_2} = \lim_{\sum_{i=1}^{n} x_i^{-\beta} \to \infty} \hat{\theta}_{EBDG_2} - \hat{\theta}_{EBDG_3} = 0$$

That is, $\lim_{\sum_{i=1}^{n} x_i^{-\beta} \to \infty} \hat{\theta}_{EBDG_1} = \lim_{\sum_{i=1}^{n} x_i^{-\beta} \to \infty} \hat{\theta}_{EBDG_2} = \lim_{\sum_{i=1}^{n} x_i^{-\beta} \to \infty} \hat{\theta}_{EBDG_3}$.

Thus, the proof of the theorem is complete.

Theorem 4.2:- E-Bayes estimator obtained under Al-Bayyati loss function follow the following results:-

i. $\hat{\theta}_{EBAB_3} < \hat{\theta}_{EBAB_2} < \hat{\theta}_{EBAB_1}$.

ii. $\lim_{\sum_{i=1}^{n} x_i^{-\beta} \to \infty} \hat{\theta}_{EBAB_1} = \lim_{\sum_{i=1}^{n} x_i^{-\beta} \to \infty} \hat{\theta}_{EBAB_2} = \lim_{\sum_{i=1}^{n} x_i^{-\beta} \to \infty} \hat{\theta}_{EBAB_3}$.

Proof (i) :- From (3.11) and (3.12)

$$\hat{\theta}_{EBAB_1} - \hat{\theta}_{EBAB_2}$$

$$= \frac{(2n+1+2c_1)}{s}\left\{\log\left(1+\frac{s}{\sum_{i=1}^{n} x_i^{-\beta}}\right)*\left(\frac{\sum_{i=1}^{n} x_i^{-\beta}}{s}+\frac{1}{2}\right)-1\right\} \tag{4.5}$$

From equation, (3.12) and (3.15) we have

$$\hat{\theta}_{EBAB_2} - \hat{\theta}_{EBAB_3} = \frac{(2n+1+2c_1)}{s}\left\{\log\left(1+\frac{s}{\sum_{i=1}^{n} x_i^{-\beta}}\right)*\left(\frac{\sum_{i=1}^{n} x_i^{-\beta}}{s}+\frac{1}{2}\right)-1\right\} \tag{4.6}$$

From (4.5) and (4.6), we get

$$\hat{\theta}_{EBAB_1} - \hat{\theta}_{EBAB_2} = \hat{\theta}_{EBAB_2} - \hat{\theta}_{EBAB_3}$$

$$= \frac{(2n+1+2c_1)}{s}\left\{\log\left(1+\frac{s}{\sum_{i=1}^{n} x_i^{-\beta}}\right)*\left(\frac{\sum_{i=1}^{n} x_i^{-\beta}}{s}+\frac{1}{2}\right)-1\right\} \tag{4.7}$$

$$\log\left(1+\frac{s}{\sum_{i=1}^{n} x_i^{-\beta}}\right) = \left(\frac{s}{\sum_{i=1}^{n} x_i^{-\beta}} - \frac{s^2}{2\left(\sum_{i=1}^{n} x_i^{-\beta}\right)^2} + \frac{s^3}{3\left(\sum_{i=1}^{n} x_i^{-\beta}\right)^3} - \cdots\right)$$

$$\hat{\theta}_{EBAB_1} - \hat{\theta}_{EBAB_2} = \frac{(2n+1+2c_1)}{s} * \left\{\frac{s^2}{12\left(\sum_{i=1}^{n} x_i^{-\beta}\right)^2} + \frac{s^3}{6\left(\sum_{i=1}^{n} x_i^{-\beta}\right)^3} - \ldots\ldots\ldots..\right\} > 0$$

$$\Rightarrow \hat{\theta}_{EBAB_3} < \hat{\theta}_{EBAB_2} < \hat{\theta}_{EBAB_1}.$$

Proof (ii) :- From equation (4.7), we have

$$\hat{\theta}_{EBAB_1} - \hat{\theta}_{EBAB_2} = \hat{\theta}_{EBAB_2} - \hat{\theta}_{EBAB_3}$$

$$= \frac{(2n+1+2c_1)}{s}\left\{\log\left(1+\frac{s}{\sum_{i=1}^{n} x_i^{-\beta}}\right)*\left(\frac{\sum_{i=1}^{n} x_i^{-\beta}}{s}+\frac{1}{2}\right)-1\right\}$$

$$\lim_{\sum_{i=1}^{n} x_i^{-\beta}\to\infty} \hat{\theta}_{EBAB_1} - \hat{\theta}_{EBAB_2} = \lim_{\sum_{i=1}^{n} x_i^{-\beta}\to\infty} \hat{\theta}_{EBAB_2} - \hat{\theta}_{EBAB_3} = 0$$

Thus the proof is complete.

Theorem 4.3:- E-Bayes estimator obtained under Minimum-expected loss function follow the following results:-

i. $\hat{\theta}_{EBME_3} < \hat{\theta}_{EBME_2} < \hat{\theta}_{EBME_1}$.

ii. $\lim_{\sum_{i=1}^{n} x_i^{-\beta}\to\infty} \hat{\theta}_{EBME_1} = \lim_{\sum_{i=1}^{n} x_i^{-\beta}\to\infty} \hat{\theta}_{EBME_2} = \lim_{\sum_{i=1}^{n} x_i^{-\beta}\to\infty} \hat{\theta}_{EBME_3}$.

Proof (i) :- From (3.16) and (3.17)

$$\hat{\theta}_{EBME_1} - \hat{\theta}_{EBME_2}$$

$$= \frac{2n-3}{s}\left\{\log\left(1+\frac{s}{\sum_{i=1}^{n} x_i^{-\beta}}\right)*\left(\frac{s+2\sum_{i=1}^{n} x_i^{-\beta}}{2s}\right)-1\right\} \tag{4.8}$$

From equation, (3.17) and (3.18) we have

$$\hat{\theta}_{EBME_2} - \hat{\theta}_{EBME_3} = \frac{2n-3}{s}\left\{\log\left(1+\frac{s}{\sum_{i=1}^{n} x_i^{-\beta}}\right)*\left(\frac{s+2\sum_{i=1}^{n} x_i^{-\beta}}{2s}\right)-1\right\} \tag{4.9}$$

From (4.8) and (4.9), we get

$$\hat{\theta}_{EBME_1} - \hat{\theta}_{EBME_2} = \hat{\theta}_{EBME_2} - \hat{\theta}_{EBME_3}$$

$$= \frac{2n-3}{s}\left\{\log\left(1+\frac{s}{\sum_{i=1}^{n} x_i^{-\beta}}\right) * \left(\frac{s+2\sum_{i=1}^{n} x_i^{-\beta}}{2s}\right) - 1\right\} \qquad (4.10)$$

$$\log\left(1+\frac{s}{\sum_{i=1}^{n} x_i^{-\beta}}\right) = \left(\frac{s}{\sum_{i=1}^{n} x_i^{-\beta}} - \frac{s^2}{2\left(\sum_{i=1}^{n} x_i^{-\beta}\right)^2} + \frac{s^3}{3\left(\sum_{i=1}^{n} x_i^{-\beta}\right)^3} - \cdots\right)$$

$$\hat{\theta}_{EBAB_1} - \hat{\theta}_{EBAB_2} = \frac{2n-3}{s} * \left\{\frac{s^2}{12\left(\sum_{i=1}^{n} x_i^{-\beta}\right)^2} + \frac{s^3}{6\left(\sum_{i=1}^{n} x_i^{-\beta}\right)^3} - \dots\dots\dots..\right\} > 0$$

$$\Rightarrow \hat{\theta}_{EBAB_3} < \hat{\theta}_{EBAB_2} < \hat{\theta}_{EBAB_1}.$$

Proof (ii) :- From equation (4.10), we have

$$\hat{\theta}_{EBME_1} - \hat{\theta}_{EBME_2} = \hat{\theta}_{EBME_2} - \hat{\theta}_{EBME_3}$$

$$\lim_{\sum_{i=1}^{n} x_i^{-\beta} \to \infty} \hat{\theta}_{EBME_1} - \hat{\theta}_{EBME_2} = \lim_{\sum_{i=1}^{n} x_i^{-\beta} \to \infty} \hat{\theta}_{EBME_2} - \hat{\theta}_{EBME_3} = 0$$

Thus the proof is complete.

The part (i) of theorem 4.1, 4.2 and 4.3 shows that with different priors (3.1) – (3.3) of the hyper parameters c and r, the corresponding E-Bayes estimates $\hat{\theta}_{EBDG_i}$, $\hat{\theta}_{EBAB_i}$ and $\hat{\theta}_{EBME_i}$ (i=1, 2, 3) are different. The part (ii) of both the theorems shows that $\hat{\theta}_{EBDG_i}$, $\hat{\theta}_{EBAB_i}$ and $\hat{\theta}_{EBME_i}$; (i=1, 2, 3) are asymptotically equivalent to each other as $\sum_{i=1}^{n} x_i^{-\beta} \to \infty$, that means $\hat{\theta}_{EBDG_i}(i = 1,2,3)$ are all close to each other when $\sum_{i=1}^{n} x_i^{-\beta}$ is sufficiently largeand $\hat{\theta}_{EBAB_i}$, $\hat{\theta}_{EBME_i}$; (i =1, 2, 3) are also close to each other .

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